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The Rise, Fall and Sustainability of Capital-Resource Economies

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Abstract

In debates about green accounting it is sometimes argued that a positive value of aggregate investments indicates that an economy is developing sustainably. Asheim (1994) and Pezzey (1994) have shown that this is wrong, using a version of the well-known Dasgupta–Heal economy (with one capital and one non-renewable resource stock) as a counterexample. Asheim's proof referred to the unproved assumptions that in such an economy a higher rate of time preference induces higher initial consumption and vice versa, and that "optimal" consumption is initially rising and then falling. Here we show that these assumptions do hold true under certain circumstances, thereby also proving some of Dasgupta and Heal's other conjectures about sustainability.

I. Introduction

Perhaps most mainstream economists nowadays take it for granted that the objective of growth and development is to maximise the present discounted value (PV) of social utility using a constant discount rate, or "rate of impatience"; see e.g. Barro and Sala-i-Martin (1995, p. 61). However, many other economists, both mainstream and "alternative", vigorously debate the alternatives to selecting this *PV-optimal* path, as we call it here. One alternative could be to maximise PV defined with a non-constant discount rate, as stressed by Asheim (1994). A special case of this corresponds to the Rawlsian "maxi-min" path, or path of maximum constant utility, first analysed by Solow (1974), and much promoted by Hartwick (1977) and subsequent literature. Or, society could obey one of several simple sustainability constraints, as surveyed by Pezzey (1989). Many of these alternatives could usefully be studied with overlapping-

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generations models, as for example in Howarth and Norgaard (1995), rather than with the representative-agent models on which we focus here. Beyond all this lie any number of possible objectives for intertemporal choice, some of which may not even select a Pareto-efficient path; see Pezzey (1997).

Comparisons among these development objectives inevitably lead to an apparently schizophrenic practice: that of examining a path which optimises one objective, from the viewpoint of another. This would make no sense if there were no debate. If society indisputably were to seek sustainability, one would not worry about its cost in terms of foregone PV. Or if society indisputably were to seek PV-optimality, one would not worry, as we do throughout this paper, whether or not PV-optimal paths are sustainable.

The latter worry was already apparent in the classic papers on growth and resources by Dasgupta and Heal (1974, hereafter DH74), Stiglitz (1974) and Solow (1974), and appeared frequently in the text by Dasgupta and Heal (1979, hereafter DH79). In DH79, the authors compared various intertemporal objectives (Chapter 9) and then compared the maxi-min and PV-optimal paths (Chapter 10). They focused on simple economies — with one human-made capital stock, one non-renewable resource stock, constant population and no technical progress — which we call *Dasgupta–Heal economies* here and define carefully below. In such an economy with diminishing-returns, Cobb-Douglas production, they claimed to find that PV-optimal consumption, and with it well-being, typically rises to a single peak, and then declines forever towards zero consumption asymptotically. Their reaction to this result was a perfect example of apparent schizophrenia, which questioned PV-optimality by the standards of intergenerational equity:

> "Of course, later generations ... suffer incredibly as a result of the initial profligacy under the Utilitarian [i.e. PV-optimal] programme" (DH79, p. 299)

However, except for a special case in DH74, they did not actually prove this claim. Nor did they prove further claims that raising the utility discount rate brings forward the peak time of the PV-optimal path, and raises the initial consumption level, eventually above the maximum sustainable level. The task we have set here is to find formal proofs for these claims. This turns out to be quite difficult, and we cover only the rather pessimistic case of a constant (mostly Cobb-Douglas) technology, Dasgupta–Heal economy. In Section II we define a Dasgupta–Heal economy, and four intuitively appealing properties about peakedness and sustainability which we suspect are true in a fairly general case. We then give such proofs as we have found: for a constant returns economy in Section III, for a Cobb-

Douglas economy in Section IV, and for a special case of the Section IV economy in Section V. Section VI concludes.

Is it worth clearing up such technical minutiae after all this time? We think so. A focus of recent academic debate about sustainable development has been on comparing intertemporal objectives and measuring progress towards them. Two prominent writers on growth and resources active in the 1970s have also contributed to this debate; see Solow (1993a, 1993b) and Dasgupta (1994). A society may be contemplating a shift to sustainability, but may be concerned whether this will cause an initial drop in well-being. This society may therefore wish to know how impatient it can be before its PV-optimal path is initially unsustainable. Asheim (1994, p. 263) has added to the "green accounting" debate by using two of DH79's unproved claims to contradict the common view that momentarily non-negative total investment (the combined value of capital investments minus resource rents) guarantees sustainability at that moment. Underpinning the original DH claims therefore seems worthwhile.

II. The Dasgupta–Heal Economy: Definitions and Statement of Properties

We define a Dasgupta–Heal economy as follows. The economy has initial (at $t=0$) stocks of man-made capital K_0 and of a non-renewable natural resource S_0. The technology is given by a production function F having as inputs capital (K) and the rate of depletion (R) of the non-renewable resource. The production function satisfies:

F: $\mathbb{R}^2_+ \to \mathbb{R}_+$ is on $\mathbb{R}^2_{++}$ strictly increasing, strictly concave and twice continuously differentiable. Furthermore, $F(K,0)=F(0,R)=0$ and for $K>0$ we have $\partial F(K,R)/\partial R \to \infty$ as $R \to 0$.

A consumption trajectory C: $[0,\infty)\to\mathbb{R}_+$ is called feasible if there are $(K,S,R):[0,\infty)\to\mathbb{R}^3_+$ such that for all $t\geq 0$

$$\dot{K}(t)=F[K(t),R(t)]-C(t), \quad K(0)=K_0 \tag{1}$$

$$\dot{S}(t)=-R(t), \quad S(0)=S_0. \tag{2}$$

Many of the assumptions in this model are, of course, considerable abstractions from reality. First, implicit in (1) are the assumptions of a constant population and labour force,[1] and a constant technology.[2] Second,

[1] Hence we will not need to distinguish between total and per capita well-being.

[2] Alternative assumptions would be that of exogenous, exponential technical progress as in Stiglitz (1974) or sub-exponential progress as in Pezzey (1994, p. 53). Either may still cause single-peakedness, though a full analysis remains for further work.

(1) assumes that the produced good can be used equally well for consumption or investment. As Weitzman (1976, p. 159) notes, this is as much an approximation to reality as any other assumption. Third, (1) assumes that capital depreciation is zero, while capital consumption ($\dot{K}<0$, with no lower bound) is possible. Indeed, this will be a key feature of the PV-optimal path in Section V. Fourth, the final condition on $\partial F/\partial R$ is one of DH74's (p. 25) two conditions for resource flow to be *essential*.[3] So, the model ignores the possible existence of a backstop technology or renewable substitutes for the natural resource as an input to production.

Next we introduce the notion of PV-optimality. Let $\delta>0$ denote the rate of pure time preference or impatience, and U an instantaneous utility function satisfying

$U:\mathbb{R}_+\to\mathbb{R}$ is on $\mathbb{R}_{++}$ strictly increasing, strictly concave, and twice continuously differentiable. Furthermore $\partial U(C)/\partial C\to\infty$ as $C\to 0$.

A consumption trajectory $C:[0,\infty)\to\mathbb{R}_+$ is called *PV-optimal* and denoted $\tilde{C}(t;\delta)$ if it is feasible and maximises the present value (PV) integral

$$W\{C\}:=\int_0^\infty U[C(t)]\,\mathrm{e}^{-\delta t}\,\mathrm{d}t$$

over all feasible consumption trajectories.

Along a PV-optimal path, C and R will always be strictly positive due to the assumptions with respect to U and F. Using Theorem 9 of Seierstad and Sydsaeter (1987, p. 381), using subscripts to denote partial derivatives and omitting the argument t, we have as necessary conditions for PV-optimality:

$$\dot{F}_R(K,R)/F_R(K,R)=F_K(K,R) \qquad \text{(Hotelling's rule)} \tag{3}$$

$$\eta(C)\dot{C}/C=F_K(K,R)-\delta \qquad \text{(Ramsey's rule)} \tag{4}$$

where $\eta(C):=-U_{CC}C/U_C\ (>0)$ is the elasticity of marginal utility.

A consumption path $C:[0,\infty)\to\mathbb{R}$ is called *sustainable* at t if it is no greater at t than the maximal constant feasible consumption level given $K(t)$ and $S(t)$, and *initially sustainable* if it is sustainable at $t=0$.[4] It is called *single-peaked* if there exists $T_{\mathrm{P}}\geq 0$ such that $\dot{C}(t)>0$ for all $t\in[0,T_{\mathrm{P}})$, and $\dot{C}(t)\leq 0$ for all $t\in[T_{\mathrm{p}},\infty)$. Finally, it is often enough to use C^δ instead of

[3] We ignore the different definition of (resource) essentiality in DH79 (p. 198), namely that "feasible consumption must necessarily decline to zero in the long run".

[4] Contrary to what one of us wrote earlier, cf. Pezzey (1989), we do not define sustainability as non-declining utility, at a point in time or over a period of time.

$\tilde{C}(t;\delta)$ to denote the PV-optimal consumption trajectory when the rate of time preference is δ. The properties which we suspect are true of the PV-optimal path C^{δ} of *most* Dasgupta–Heal economies are as follows:

Property 1. *C^{δ} is single-peaked for all $\delta > 0$.*
Property 2. *For low enough δ, C^{δ} is initially sustainable.*
  Property 2a. *For low enough δ, C^{δ} is initially rising.*
Property 3. *For high enough δ, C^{δ} is initially falling.*
  Property 3a. *For high enough δ, C^{δ} is initially unsustainable.*
Property 4. *If the peak time T_P is positive, it is decreasing in δ.*

These properties are illustrated by Figure 1, where C_0^m is the maximum consumption level at $t = 0$. Properties 2a and 3a are indented because they follow respectively from Properties 1 and 2, and from Properties 1 and 3. In turn this is because both the PV-optimal and maximin paths are Pareto efficient, and so must intersect one another. The precise properties proved, claimed or implied by DH74 and DH79, and proved in Sections III–V below, are given in Table 1.

The stated limitations on the production functions are important here. Moreover, the non-renewability and essentiality of the resource base for R, and the absence of exogenous technical progress, will be crucial in our proofs of single-peakedness in both Sections III and IV. Together they eventually drive the marginal product of capital $F_K(K, R)$ below the rate of

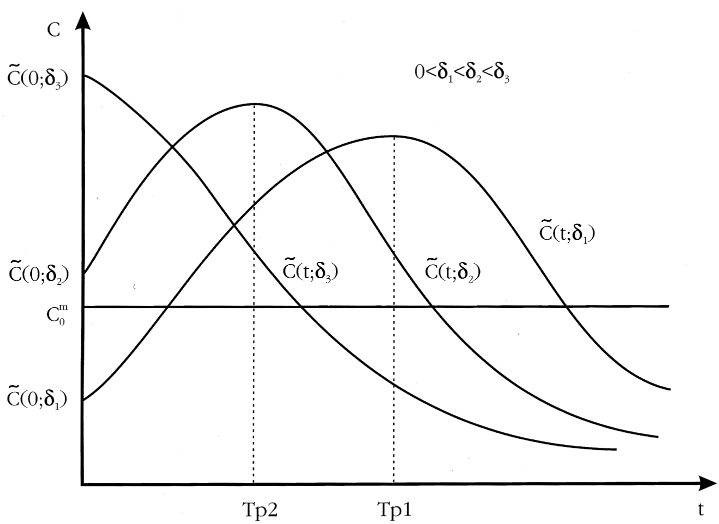

Fig. 1. Single-peakedness and initial sustainability

impatience δ, causing PV-optimal consumption to decline according to the Ramsey equation (4). But if there instead is a flow $\bar{R}$ bounded away from zero which can be maintained forever by renewable sources, and/or technical progress ($F_K(K, R, t)$ rather than $F_K(K, R)$), then it isno longer inevitable that F_K will be driven below δ. One would instead have to argue as follows that consumption must decline below current levels. Technical progress is bounded; current resource flows are unsustainable ($R \gg \bar{R}$); huge increases in capital would therefore be needed to maintain consumption; but such decreased R and increased K would still force F_K below δ and keep it there. However, this is an *empirical* argument, beyond the scope of this paper.

III. Single-Peakedness in a Constant Returns to Scale, Dasgupta–Heal Economy

In this section, it is assumed that F displays constant returns to scale (Assumption 1 in Table 1), in addition to the assumptions made in the

Table 1. *Summary of claims and results about single-peakedness and initial sustainability in a Dasgupta–Heal economy*

	Type of Dasgupta–Heal economy		
	Assumption 1: General production function with constant returns	*Assumption 2:* Cobb–Douglas[a] production $F = K^{\alpha_1}R^{\alpha_2}$ with $0 < \alpha_2 < \alpha_1 < \alpha_1 + \alpha_2 \leq 1$; isoelastic utility $U = C^{1-\eta}/(1-\eta)$	
		Constant returns $\alpha_1 + \alpha_2 = 1$	Diminishing returns $\alpha_1 + \alpha_2 < 1$
Property 1	—	Proved by DH74 (p. 17)	Claimed by DH79 (p. 299)
Properties 2–3	—	—	Claimed by DH79 (p. 308)
Property 4	—	—	Claimed by DH79 (p. 299)
Property 1	Proved in our Section III	Proved in our Section IV	
Properties 2–3	—		
Property 4	—	Proved for special case $\alpha_1 = \eta$ in our Section V	—

[a] Since the elasticity of substitution between capital and natural resources is of vital interest to sustainability, one should ideally go beyond the unit elasticity of the Cobb–Douglas form, for example to a constant elasticity of substitution. However, the analytical methods developed below are intractable for the CES form.

preceding section. The following then holds (we omit the index δ and the argument t where possible):

Proof of Property 1 under Assumption 1. Let C be PV-optimal and let (K, S, R) be the corresponding paths of the capital stock, resource stock and resource use. Since $R > 0$ always and in view of constant returns to scale, $x := K/R$ and $f(x) := F(x, 1)$ are well defined. Hence $F_K(K, R) = f'(x) > 0$, and also $F_R(K, R) = f(x) - xf'(x) > 0$. Moreover, due to strict concavity we have $f''(x) < 0$. In this case, Hotelling's rule (3) reads

$$-x\dot{x}f''(x)/[f(x) - xf'(x)] = f'(x).$$

Hence $\dot{x} > 0$ and $f'(x)$ is monotonically decreasing over time. Ramsey's rule (4) reads

$$\eta(C)\dot{C}/C = f'(x) - \delta.$$

Therefore, all the proof requires is that $f'(x) - \delta$ is negative for all t sufficiently large. Suppose to the contrary that $f'(x(t)) - \delta \geq 0$ for all $t \geq 0$ for all $t \geq 0$. Then it follows from Ramsey's rule that C is bounded below by some $\bar{C} > 0$. If also x is bounded above by some $M > 0$ for all $t \geq 0$, then

$$\dot{K} = RF(x, 1) - C \leq RF(M, 1) - \bar{C}.$$

But if this holds for all $t \geq 0$, then either K or S will become negative, which is not allowed; hence x is not bounded above. Hence if we take M large enough that $F(1, 1/M) < \delta$ (which is possible since $F(1, 0) = 0$), we can find t such that $x(t) > M$, and then arrive at the following contradiction:

$$0 < F_R = [F(K, R) - KF_K]/R = x[F(1, 1/x) - F_K] \leq x[F(1, 1/M) - \delta] < 0.$$

This proves that $f'(x(t)) - \delta < 0$ for all t sufficiently large, as required. □

However, in the above case we have not been able to prove anything about the effect of the utility discount rate δ on the initial slope or sustainability of the PV-optimal consumption path (Properties 2–3), or the peak time (Property 4). We turn now to the Cobb–Douglas economy, where we can prove Properties 2–3 (but not 4).

IV. Single-Peakedness and Initial Sustainability in a Cobb–Douglas, Dasgupta–Heal Economy

The additional assumptions for the Dasgupta–Heal economy here are that the utility function is isoelastic, and that the production function is Cobb–Douglas with constant or diminishing returns to capital and resource flow:

$$U(C) = C^{1-\eta}/(1-\eta), \qquad \eta > 0, \quad \eta \neq 1 \tag{5}$$

$$F(K, R) = K^{\alpha_1} R^{\alpha_2}; \qquad 0 < \alpha_2 < \alpha_1 < \alpha_2 \leq 1. \tag{6}$$

The requirement that $\alpha_2 < \alpha_1$ is Solow's (1974) condition for a non-zero maximin path to exist. Constant returns ($\alpha_1 + \alpha_2 = 1$) were considered by DH74 (p. 17), whose equation (1.37) establishes Property 1 only. Diminishing returns ($\alpha_2 + \alpha_2 < 1$) were considered by DH79 (p. 298). (They also imposed a tighter restriction on U, that $\eta > 1$, but this is irrelevant here.) Close comparison of DH79 (p. 299) with our Properties 1–4, allowing for differences in terminology and notation, indicates that for diminishing returns, DH79 said "[i]t can be shown that" Properties 1 and 4 are true. Their Diagram 10.3 (p. 299), very similar to our Figure 1, implied Properties 2 and 3; they also claimed Properties 2a and 3 (p. 308). Asheim (1994, p. 262) used Properties 2 and 3a to deduce by continuity that a δ exists in this case which makes initial PV-optimal consumption rising and just sustainable, i.e., equal to initial maximin consumption. Hence, non-negative total (aggregate) investment does not guarantee sustainability.

But DH79 did not actually prove their claims.[5] The limit of their analysis was the pair of non-autonomous equations for $(d/dt)[R^{(1-\alpha_2)/\alpha_1}]$ and $\dot{K}$, (in our notation), which they noted are "difficult to dissect in detail" (p. 302). We now show Properties 1–3 to be true, for both constant and diminishing returns, thus advancing beyond a more heuristic treatment of the Cobb-Douglas case by Hamilton (1995, p. 406). The proofs fall into a number of sub-cases, one of which is quite tedious and is therefore omitted.

To cope with diminishing returns, we have to abandon the method of Section III. Instead we compute the asymptotic steady state, and deduce the pre-steady-state behaviour of the system from a phase diagram.[6] The current value Hamiltonian of the PV-maximisation problem is

$$\mathscr{H}(K, S, C, R, \pi_K, \pi_S) = C^{1-\eta}/(1-\eta) + \pi_K(K^{\alpha_1}R^{\alpha_2} - C) + \pi_S(-R);$$

and applying the maximum principle shows that the costate variables π_K and π_S, corresponding to the capital and resource stocks on the PV-optimal path, obey the following equations (dropping the time argument to save clutter):

[5] According to DH79 (p. 321), their "exposition [is] based on Dasgupta (1977a)", an unpublished draft which we have been unable to obtain. However, apart from the date, the citation of Dasgupta (1977a) given in DH79 is the same as Dasgupta (1982) given here, and the latter contains no more proof of Properties 1–4 than does DH79.
[6] In this we follow Stiglitz (1974, p. 134). He used a more general production function which also included population growth and technical progress, but a more restrictive, logarithmic utility function. See also Withagen (1990).

$$\left.\begin{array}{ll} \partial\mathcal{H}/\partial C = 0: & C^{-\eta} = \pi_K \\ \partial\mathcal{H}/\partial K = -\dot{\pi}_K + \delta\pi_K: & -\dot{\pi}_K + \delta\pi_K = \pi_K\alpha_1 K^{\alpha_1 - 1}R^{\alpha_2} \\ \partial\mathcal{H}/\partial R = 0: & \pi_K\alpha_2 K^{\alpha_1}R^{\alpha_2 - 1} = \pi_S \\ \partial\mathcal{H}/\partial S = -\dot{\pi}_S + \delta\pi_S: & -\dot{\pi}_S + \delta\pi_S = 0 \end{array}\right\} \quad (7)$$

Following Stiglitz (1974) we define $\beta := F/K$, $\zeta := C/K$; and $g_y :=$ the growth rate $\dot{y}/y$ for any variable y. The above necessary conditions (7) for PV-optimality and the production relationship (1) then reduce to

$$\begin{array}{llll} g_\beta & -\alpha_2 g_R & +(1-\alpha_1)g_K & = 0 \\ & -(1-\alpha_2)g_R & +\alpha_1 g_K & = \alpha_1\beta \\ \eta g_\zeta & & +\eta g_K & = \alpha_1\beta - \delta \\ & & g_K & = \beta - \zeta \end{array} \quad (8)$$

which yield

$$g_\zeta = \zeta + (\alpha_1/\eta - 1)\beta - \delta/\eta \quad (9)$$

$$g_\beta = [(1-\alpha_1-\alpha_2)/(1-\alpha_2)]\zeta - (1-\alpha_1)\beta \quad (10)$$

$$g_C = (\alpha_1\beta - \delta)/\eta \quad (11)$$

$$g_R = -\alpha_1\zeta/(1-\alpha_2). \quad (12)$$

Properties 1–3 can now be considered under two separate cases.

THE CASE $\alpha_1 \geq \eta$

If $\alpha_1 > \eta$ and $\alpha_1 + \alpha_2 < 1$ (diminishing returns), the phase diagram in (β, ζ)-space is as depicted in Figure 2. From (10), the slope of the $g_\beta = 0$ line equals $(1-\alpha_1-\alpha_2)/(1-\alpha_1)(1-\alpha_2)$, which is less than 1, the slope of the $g_K = 0$ line from (8). In the $\alpha_1 + \alpha_2 = 1$ (constant returns) case, the $\gamma_\beta = 0$ line becomes the horizontal axis; and if $\alpha_1 = \eta$, the $\gamma_\zeta = 0$ line becomes the vertical line $\zeta = \delta/\alpha_1$; but neither special case affects any of the following analysis. From (8)–(11) the $g_K = 0$, $g_C = 0$ and $g_\zeta = 0$ lines all intersect at $(\delta/\alpha_1, \delta/\alpha_1)$, and the asymptotic equilibrium point E has coordinates:

$$\beta_\infty = \delta(1-\alpha_2-\alpha_2)/\alpha_1[1-\alpha_1-\alpha_2(1-\eta)] \quad (13)$$

$$\zeta_\infty = \delta(1-\alpha_1)(1-\alpha_2)/\alpha_1[1-\alpha_1-\alpha_2(1-\eta)]. \quad (14)$$

Along a PV-optimal trajectory, $(\beta(t), \xi(t)) \to (\beta_\infty, \zeta_\infty)$ as $t \to \infty$, i.e., point E, because otherwise either ζ will become negative, which is not allowed, or K will become negative (since $\beta \to \infty$ and $\dot{C}/C \to \infty$), which is also not

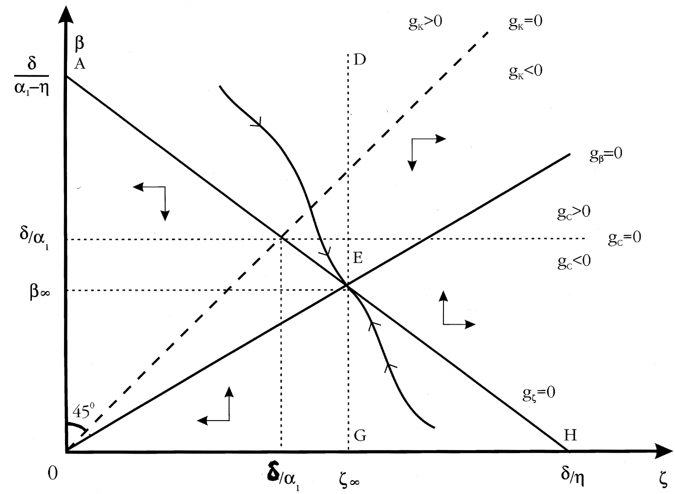

Fig. 2. Phase diagram for the diminishing returns, Cobb–Douglas case with $\alpha_1 > \eta$

allowed. E is a saddle point, with approach to equilibrium possible only in the "feasible sectors" AED and GEH (the latter does not exist in the constant returns case, and the sectors are just vertical lines in the $\alpha_1 = \eta$ case). From (13) we have $\beta_\infty < \delta/\alpha_1$, the value of β for which $g_C = 0$. We can then give:

Proof of Property 1 under Assumption 2, provided that $\alpha_1 \geq \eta$. The PV-optimal path either starts above the $g_C = 0$ line in a zone of rising consumption, and then passes after finite time to remain in a zone of falling consumption; or it starts and stays below the $g_C = 0$ line in a zone of falling consumption. □

Proof of Property 2 under Assumption 2, provided that $\alpha_1 \geq \eta$. Solow (1974) showed that, given the parameter restriction $\alpha_1 > \alpha_2$, the initial maximin consumption level is

$$C_0^{\mathrm{m}} = (1-\alpha_2)\,[(\alpha_1-\alpha_2)^{\alpha_2} S_0^{\alpha_2} K_0^{\alpha_1-\alpha_2}]^{1/(1-\alpha_2)}. \tag{15}$$

As a property solely of the production side of the economy, naturally C_0^{m} is independent of impatience δ. Suppose δ is small enough so that $C_0^{\mathrm{m}}/K_0 > \delta/\eta$. Then $C^\delta(0) < C_0^{\mathrm{m}}$ because otherwise $\zeta(0) = C^\delta(0)/K_0 > \delta/\eta$, which would put the whole path outside the feasible sectors for a PV-optimal trajectory, a contradiction. □

Proof of Property 3 under Assumption 2, provided that $\alpha_1 \geq \eta$. Suppose not, i.e., for all $\varepsilon > 0$ there exists a $\delta < \varepsilon$ such that $\dot{C}^\delta(0) \geq 0$. From the phase diagram, starting on or above the $g_C = 0$ line means $\beta(0) \geq \delta/\alpha_1$ and $\zeta(t) \leq \zeta_\infty$ for all t. Hence, using (12) and (14) in turn,

$$R(0) = [\beta(0) K_0^{1-\alpha_1}]^{1/\alpha_2} \geq (\delta K_0^{1-\alpha_1}/\alpha_1)^{1/\alpha_2}$$

and

$$g_R \geq -\alpha_1 \zeta_\infty/(1-\alpha_2) = -\delta(1-\alpha_1)/[1-\alpha_1-\alpha_2(1-\eta)].$$

Hence $R(t) \geq (\delta K_0^{1-\alpha_1}/\alpha_1)^{1/\alpha_2} e^{-\delta(1-\alpha_1)t/[1-\alpha_1-\alpha_2(1-\eta)]}$.

But the integral from time zero to infinity of this lower bound of R is $\delta^{(1/\alpha_2)-1}$ times some constant independent of δ. Since $1/\alpha_2 > 1$, this exceeds the available resource stock S_0 for large enough δ, a contradiction. □

THE CASE $\alpha_1 < \eta$

This is illustrated for diminishing returns by the phase diagram in Figure 3. We can then show that Properties 1–3 still hold, as follows:

Proof of Property 1 under Assumption 2, provided that $\alpha_1 < \eta$. This is trivial, since the phase diagram shows that the PV-optimal path either passes from above the $g_C = 0$ line to below it, or starts and stays below it. □

Note also that the $g_\zeta = 0$ line is upward sloping, in fact with slope $\eta/(\eta - \alpha_1) > 1$. However, the $g_\beta = 0$ line has slope less than 1 (it is the ζ-axis, in the constant returns case). So the two lines still cross, which means that ζ decreases on a single-peaked PV-optimal path. Compared to the $\alpha_1 \geq \eta$

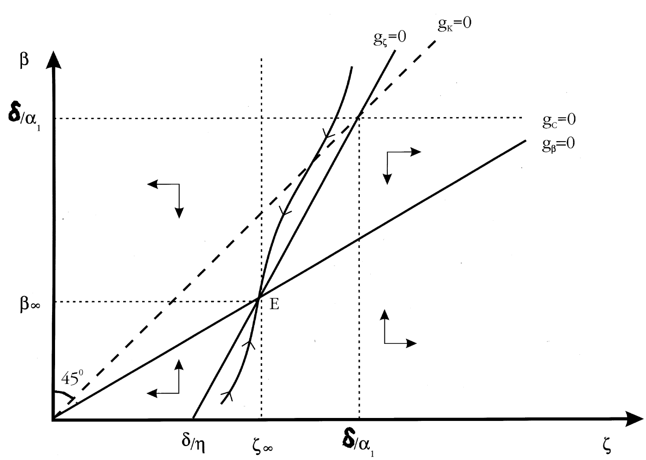

Fig. 3. Phase diagram for the diminishing returns, Cobb–Douglas case with $\alpha_1 < \eta$

case, the feasible sectors of the PV-optimal path are now bounded by $\zeta \geq \delta/\eta$ rather than by $\zeta \leq \delta/\eta$.

Proof of Property 2 under Assumption 2, provided that $\alpha_1 < \eta$. This is easy if $\eta \geq (1-\alpha_2)/(\alpha_1-\alpha_2)$, since in this case DH79 (pp. 303–7) calculated an exact solution for zero discounting ($\delta = 0$), showing that $C^0(0) < C_0^{\mathrm{m}}$. Hence, by the continuity of the variables with respect to the parameters, $C^\delta(0) \leq C_0^{\mathrm{m}}$ for δ sufficiently small. If, however, $\eta < (1-\alpha_2)/(\alpha_1-\alpha_2)$, the proof is much more complicated. It is available on request. □

Proof of Property 3 under Assumption 2, provided that $\alpha_1 < \eta$. Suppose $\dot{C}^\delta(0) \geq 0$. From Figure 3, starting on or above the $g_C = 0$ line means there is a $t' \geq 0$ such that $\beta(t') \geq \delta/\alpha_1$ and $\zeta(t') \leq \delta/\alpha_1$. So $\zeta(t) \leq \delta/\alpha_1$ for all $t \geq t'$. The rest of the proof then follows from the $\alpha_1 \geq \eta$ case. □

V. Monotone Peak Time in a Special Case

We have yet to prove that greater impatience δ brings forward the peak time T_P (Property 4). We now do this for the special case of $\alpha_1 = 1-\alpha_2 =: \alpha = \eta$ in the constant returns, Cobb–Douglas case in DH74 (pp. 11 and 17). That is, the production elasticity of capital coincides with the elasticity of marginal utility. For constant returns we have $f(x) = x^\alpha$ (recall that $x = K/R$), and Hotelling's rule (3) takes the form $\dot{x} = x^\alpha$. We can then give:

Proof of Property 4 under Assumption 2, provided that $\alpha_1 = 1-\alpha_2 =: \alpha = \eta$. The $\alpha = \eta$ assumption reduces Ramsey's rule (4) to $\alpha\dot{C}/C = \alpha x^{\alpha-1} - \delta = \alpha\dot{x}/x - \delta$. This integrates to

$$C(t)/C(0) = [x(t)/x(0)]\mathrm{e}^{-(\delta/\alpha)t}, \tag{16}$$

From (1.18) in DH74, the Hotelling rule in a Cobb–Douglas economy (with unit elasticity of substitution) is $\dot{x}/x = \dot{K}/K - \dot{R}/R = (\dot{K} - \dot{R}x)/Rx = f(x)/x = F/Rx$, whence from (16)

$$\dot{R} = (\dot{K} - F)/x = -C/x = [C(0)/x(0)]\mathrm{e}^{-(\delta/\alpha)t}. \tag{17}$$

Integrating this and using the fact that $S(t) \to 0$ as $t \to \infty$, we find

$$R(t) = (\delta/\alpha)S_0\mathrm{e}^{(-\delta/\alpha)t}.$$

Hence $x(0) = K(0)/R(0) = \alpha K_0/\delta S_0$. Inserting this, and (1.36) from DH74, $x(t) = [(1-\alpha)t + x_0^{(1-\alpha)}]^{1/(1-\alpha)}$ (the integration of $\dot{x} = x^\alpha$), into $C = -x\dot{R}$, gives

$$C^\delta(t) = (\delta^2 S_0/\alpha^2)\,[(1-\alpha)t + (\alpha K_0/\delta S_0)^{1-\alpha}]^{1/(1-\alpha)}\mathrm{e}^{(-\delta/\alpha)t}. \tag{18}$$

As far as we know, this is the first explicit expression to be found for the optimal consumption path of a constant technology, capital-resource economy. Equations (15) and (18) mean that for $\delta = \alpha^2[(2\alpha - 1)S_0/K_0]^{(1-\alpha)/\alpha}$, C^δ is initially just sustainable but rising, which gives an exact counterexample for use in Proposition 3 of Asheim (1994).

Finally, provided $\alpha/\delta > (\alpha K_0/\delta S_0)^{1-\alpha}$,[7] the peak time is:

$$T_P = [\alpha/\delta - (\alpha K_0/\delta S_0)^{1-\alpha}]/(1-\alpha) > 0 \tag{19}$$

which is decreasing in δ, as required.

Exploring this special case by spreadsheet simulations can be very instructive, and a full set of formulae is given in an earlier version of this paper; cf. Pezzey and Withagen (1995).

VI. Conclusions

Sections III and IV have shown sufficient conditions for what we call Properties 1–3 to be true for a certain "Dasgupta–Heal" type of capital-resource economy, as defined precisely in Section II. The Properties concern the single-peakedness and initial sustainability of the PV-optimal (present-value-maximising) development path of the economy. They are driven by the way in which resource non-renewability and constant technology in the economy together drive down the profitability of capital investment over time. Their intuitive appeal makes us suspect that they are true for a wide range of economies, but all we have been able to prove about the Properties may be summarised by:

Proposition 1. *For any utility discount rate $\delta > 0$, the economy's PV-optimal consumption path C^δ is single-peaked if either (a) the production function F has constant returns to scale; or (b) F is Cobb–Douglas with diminishing returns and greater elasticity of capital α_1 than elasticity of resources α_2, and the utility function U is isoelastic.*

Proposition 2 (3). *For δ small (large) enough, C is initially sustainable and hence rising (falling and hence unsustainable) if F is Cobb–Douglas with constant or diminishing returns to scale and $\alpha_1 > \alpha_2$, and U is isoelastic.*

We have also been able to show that, in a special case, the peak time is monotone with respect to changes in δ.

The importance of these results is mainly technical. They prove and extend a number of unproved claims in Dasgupta and Heal (1979) about

[7]Notice that this requires a large enough initial resource stock S_0, or, contrary to DH74 (p. 17), a *small* enough initial capital stock K_0.

the PV-optimal consumption paths of such economies. They thus prove an otherwise unsupported counterexample, which Asheim (1994) used to show that momentarily non-negative total investment in capital and resource stocks on a PV-optimal path does not guarantee its sustainability at that moment. More generally though, our results underline findings for steady-state, PV-optimal economies, for example in Stiglitz (1974) and Pezzey (1989), that human impatience (a high utility discount rate) harms prospects for sustainability. Moreover, one should not worry that this approach to studying the sustainability of PV-optimal economies is self-contradictory. The current debate about intergenerational equity demands that we examine the outcomes of one possible development objective by the standards of other possible objectives. PV-optimality may not remain (if it ever was) an overriding social goal once its full implications, which studies such as ours help to reveal, become clear.

Our analysis used at least eight physically unrealistic assumptions. Three of them (no technical progress, no resource renewability, and an infinite time horizon, i.e. greater than the life of the sun) are too pessimistic about sustainability. Allowing for technical progress would be particularly desirable, since Stiglitz (1974) has shown how sufficient technical progress will remove the conflict between PV-optimality and sustainability. But three other assumptions (no capital depreciation, no population growth, and production functions which allow an unbounded ratio of the value of output to the mass of resource inputs) are too optimistic. The seventh and eighth assumptions — consumable capital, no uncertainty — cannot be so neatly categorised. In further work it would be an interesting, though inherently empirical task for reasons explained earlier, to explore the effect on single-peakedness and initial sustainability of changing these assumptions.

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